① Derivation of the oscillation probability

We start with Dirac eq:

$$i\frac{\partial\psi}{\partial t} = (\vec{\alpha}\cdot\vec{p} + \beta m)\psi = V\,\mathrm{diag}(E, E, -E, -E)V^{-1}\psi, \quad \psi = \begin{pmatrix}\psi^{(1)}\\ \psi^{(2)}\\ \psi^{(3)}\\ \psi^{(4)}\end{pmatrix}$$

with $E \equiv \sqrt{\vec{p}^{\,2} + m^2}$

Redefining $V^{-1}\psi \to \psi$, and taking only the first component, we get

$$i\frac{\partial\psi^{(1)}}{\partial t} = E\psi^{(1)} \qquad \text{where } E \equiv \sqrt{\vec{p}^{\,2} + m^2}$$

Now we introduce the two mass eigenstates (we drop the superscript "(1)" here after):

$$i\frac{\partial\nu_1}{\partial t} = E_1\nu_1 \qquad E_1 \equiv \sqrt{\vec{p}^{\,2} + m_1^2}$$

$$i\frac{\partial\nu_2}{\partial t} = E_2\nu_2 \qquad E_2 \equiv \sqrt{\vec{p}^{\,2} + m_2^2}$$

Combining these two, we get

$$i\frac{\partial}{\partial t}\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix} = \begin{pmatrix}E_1 & 0\\ 0 & E_2\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix}$$

This can be easily solved as:

$$\begin{pmatrix}\nu_1(t)\\ \nu_2(t)\end{pmatrix} = \exp\left[-it\begin{pmatrix}E_1 & 0\\ 0 & E_2\end{pmatrix}\right]\begin{pmatrix}\nu_1(0)\\ \nu_2(0)\end{pmatrix}$$

If the flavor eigenstates are related to the mass eigenstates by

$$\begin{pmatrix}\nu_\mu(t)\\ \nu_\tau(t)\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\nu_1(t)\\ \nu_2(t)\end{pmatrix} = U\begin{pmatrix}\nu_1(t)\\ \nu_2(t)\end{pmatrix}, \quad U \equiv \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

then

$$\begin{pmatrix}\nu_\mu(t)\\ \nu_\tau(t)\end{pmatrix} = U\exp\left[-it\begin{pmatrix}E_1 & 0\\ 0 & E_2\end{pmatrix}\right]\underbrace{\begin{pmatrix}\nu_1(0)\\ \nu_2(0)\end{pmatrix}}_{U^{-1}\begin{pmatrix}\nu_\mu(0)\\ \nu_\tau(0)\end{pmatrix}}$$

$$= U\exp\left[-it\begin{pmatrix}E_1 & 0\\ 0 & E_2\end{pmatrix}\right]U^{-1}\begin{pmatrix}\nu_\mu(0)\\ \nu_\tau(0)\end{pmatrix}$$

Thus, the probability amplitude $A(\nu_\mu \to \nu_\tau)$ for ν_μ to transform from ν_μ to ν_τ after travelling for the time interval t is

$$A(\nu_\mu \to \nu_\tau) = (0,1)\, U \underbrace{\exp\left[-it\begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}\right]}_{\exp[-it\,\mathrm{diag}(E_1,E_2)] = \mathrm{diag}(e^{-iE_1t},\, e^{-iE_2t})} U^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \left[U\,\mathrm{diag}(e^{-iE_1t},\, e^{-iE_2t})\, U^{-1}\right]_{2,1} \leftarrow \text{(2,1) component of 2x2 matrix}$$

Here we note that

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = e^{-i\sigma_2\theta}, \quad \exp[-it\,\mathrm{diag}(E_1,E_2)]$$

$$= \exp\Big(\underbrace{-it\,\frac{E_1+E_2}{2}\mathbb{1}}_{\text{gives only a phase}} + \underbrace{it\,\frac{E_2-E_1}{2}\sigma_3}_{e^{\frac{i}{2}\Delta E\,\sigma_3 t}}\Big), \qquad E_2-E_1 = \Delta E$$

$$e^{\frac{i}{2}\Delta E t\sigma_3} = \mathbb{1}\cos\left(\frac{\Delta Et}{2}\right) + i\sigma_3\sin\left(\frac{\Delta Et}{2}\right)$$

So

$$e^{-i\sigma_2\theta}\, e^{\frac{i}{2}\Delta E t\sigma_3}\, e^{i\sigma_2\theta}$$

$$= e^{-i\sigma_2\theta}\left[\mathbb{1}\cos\frac{\Delta Et}{2} + i\sigma_3\sin\left(\frac{\Delta Et}{2}\right)\right]e^{i\sigma_2\theta} \quad \leftarrow \text{use the identity } \{\sigma_2, \sigma_3\} = 0$$

$$= \mathbb{1}\cos\frac{\Delta Et}{2} + i\sin\left(\frac{\Delta Et}{2}\right)\underbrace{e^{-i\sigma_2\theta}\,\sigma_3\, e^{i\sigma_2\theta}}_{\sigma_3 e^{+i\sigma_2\theta}\, e^{+i\sigma_2\theta}}$$

$$e^{-i\sigma_2\theta}\sigma_3 = (\mathbb{1}\cos\theta - i\sigma_2\sin\theta)\sigma_3 = \sigma_3(\mathbb{1}\cos\theta + i\sigma_2\sin\theta) = \sigma_3 e^{+i\sigma_2\theta}$$

$$= \sigma_3 e^{2i\sigma_2\theta}$$

$$= \sigma_3(\mathbb{1}\cos 2\theta + i\sigma_2\sin 2\theta)$$

$$= \sigma_3\cos 2\theta + \sigma_1\sin 2\theta$$

$$= \mathbb{1}\cos\frac{\Delta Et}{2} + i\sin\left(\frac{\Delta Et}{2}\right)(\sigma_3\cos 2\theta + \sigma_1\sin 2\theta)$$

Therefore, we get

$$A(\nu_\mu \to \nu_\tau) = \left[e^{-\frac{i}{2}t(E_1+E_2)}\left\{\cancel{\mathbb{1}\cos\frac{\Delta Et}{2}} + i\sin\frac{\Delta Et}{2}(\cancel{\sigma_3\cos 2\theta} + \sigma_1\sin 2\theta)\right\}\right]_{2,1}$$

$$= e^{-\frac{i}{2}t(E_1+E_2)}\, i\sin\frac{\Delta Et}{2}\sin 2\theta$$

$$P(\nu_\mu \to \nu_\tau) = |A(\nu_\mu \to \nu_\tau)|^2 = \left|e^{-\frac{i}{2}t(E_1+E_2)}\, i\sin\frac{\Delta Et}{2}\sin 2\theta\right|^2 = \sin^2 2\theta\,\sin^2\left(\frac{\Delta Et}{2}\right)$$

$$\Delta E = E_2 - E_1 = \sqrt{\vec{p}^2 + m_2^2} - \sqrt{\vec{p}^2 + m_1^2} \simeq |\vec{p}| + \frac{m_2^2}{2|\vec{p}|} - \left(|\vec{p}| + \frac{m_1^2}{2|\vec{p}|}\right) = \frac{1}{2|\vec{p}|}(m_2^2 - m_1^2) = \frac{1}{2E}\Delta m^2$$

$$\frac{\Delta Et}{2} \simeq \frac{\Delta m^2}{4E}t \Rightarrow \frac{\Delta m^2}{4E}ct = \frac{\Delta m^2 L}{4E} \;//$$

② The oscillation probability for the solar neutrino

The case of the solar neutrino

We have take into account the matter effect, which is proportional to the electron density and does depend on the time (= the position of the neutrino)

$$i\frac{\partial}{\partial t}\begin{pmatrix}\nu_e(t)\\ \nu_\mu(t)\end{pmatrix} = \underbrace{\left[U\begin{pmatrix}E_1 & 0\\ 0 & E_2\end{pmatrix}U^{-1} + \begin{pmatrix}A(t) & 0\\ 0 & 0\end{pmatrix}\right]}\begin{pmatrix}\nu_e(t)\\ \nu_\mu(t)\end{pmatrix}$$

$$\equiv \tilde{U}(t)\begin{pmatrix}\hat{E}_1(t) & 0\\ 0 & \hat{E}_2(t)\end{pmatrix}\tilde{U}^{-1}(t)$$ ($\because$ A hermitian matrix can always be diagonalized)

Now we assume the adiabatic condition:

$$\left|\frac{d}{dt}\tilde{U}(t)_{jk}\right| \ll \left|\tilde{E}_2(t) - \tilde{E}_1(t)\right|$$ (← the derivative of $\tilde{U}(t)$ can be always ignored)

Then we have

$$i\frac{d}{dt}\left[\tilde{U}^{-1}(t)\begin{pmatrix}\nu_e(t)\\ \nu_\mu(t)\end{pmatrix}\right] = \begin{pmatrix}\tilde{E}_1(t) & 0\\ 0 & \tilde{E}_2(t)\end{pmatrix}\left[\tilde{U}^{-1}(t)\begin{pmatrix}\nu_e(t)\\ \nu_\mu(t)\end{pmatrix}\right]$$

which can be easily solved as

$$\tilde{U}^{-1}(t)\begin{pmatrix}\nu_e(t)\\ \nu_\mu(t)\end{pmatrix} = \exp\left[-i\begin{pmatrix}\int_0^t \tilde{E}_1(t')dt' & 0\\ 0 & \int_0^t \tilde{E}_2(t')dt'\end{pmatrix}\right]\tilde{U}^{-1}(0)\begin{pmatrix}\nu_e(0)\\ \nu_\mu(0)\end{pmatrix}$$

$$\longrightarrow \begin{pmatrix}\nu_e(t)\\ \nu_\mu(t)\end{pmatrix} = \tilde{U}(t)\exp\left[-i\begin{pmatrix}\int_0^t \tilde{E}_1(t')dt' & 0\\ 0 & \int_0^t \tilde{E}_2(t')dt'\end{pmatrix}\right]\tilde{U}^{-1}(0)\begin{pmatrix}\nu_e(0)\\ \nu_\mu(0)\end{pmatrix}$$

Here $t=0$ stands for the time of production at the centor of the Sun.

Therefore, the probability amplitude is given by

$$A(\nu_e\to\nu_e) = \left[\tilde{U}(t)\exp\left[-i\begin{pmatrix}\int_0^t \tilde{E}_1(t')dt' & 0\\ 0 & \int_0^t \tilde{E}_2(t')dt'\end{pmatrix}\right]\tilde{U}^{-1}(0)\right]_{1,1}$$ ← (1,1) component

$$= \sum_{j=1}^{2}\left(\tilde{U}(t)\right)_{1j}\exp\left\{-i\int_0^t \tilde{E}_j(t')dt'\right\}\left(\tilde{U}^{-1}(0)\right)_{j1}$$

$$\left(\tilde{U}^{-1}(0)\right)_{j1} = \left(\tilde{U}^{\dagger}(0)\right)_{j1} = \left(\tilde{U}(0)\right)^{*}_{1j}$$

$$= \sum_{j=1}^{2}\left(\tilde{U}(t)\right)_{1j}\left(\tilde{U}(0)\right)^{*}_{1j}\exp\left\{-i\int_0^t \tilde{E}_j(t')dt'\right\}$$

Thus the probability $P(\nu_e \to \nu_e)$ for solar neutrino is given by

$$P(\nu_e \to \nu_e) = |A(\nu_e \to \nu_e|^2$$

$$= \left| \sum_{j=1}^{2} (\tilde{U}(t))_{1j} (\tilde{U}(0))^*_{1j} \exp\{-i\int_0^t \tilde{E}_j(t')dt'\} \right|^2$$

$$= \sum_{j=1}^{2} (\tilde{U}(t))_{1j} (\tilde{U}(0))^*_{1j} \exp\{-i\int_0^t \tilde{E}_j(t')dt'\}$$

$$\times \sum_{k=1}^{2} (\tilde{U}(t))^*_{1k} (\tilde{U}(0))_{1k} \exp\{+i\int_0^t \tilde{E}_k(t'')dt''\}$$

$$= \sum_{j,k} (\tilde{U}(t))_{1j} (\tilde{U}(t))^*_{1k} (\tilde{U}(0))^*_{1j} (\tilde{U}(0))_{1k}$$

$$\times \exp\{-i\int_0^t \Delta\tilde{E}_{jk}(t')dt'\} \quad \text{with } \Delta\tilde{E}_{jk}(t') = \tilde{E}_j(t') - \tilde{E}_k(t')$$

In the case of the solar neutrino, the path length $L = ct$ or the time interval t is so large that rapid oscillation vanishes: $\exp\{-i\int_0^t \Delta\tilde{E}_{jk}(t')dt'\} \longrightarrow \delta_{jk}$

Therefore, we get

$$P(\nu_e \to \nu_e) = \sum_{j=1}^{2} |(\tilde{U}(t))_{1j}|^2 \, |(\tilde{U}(0))_{1j}|^2$$

At the end point, the matter density can be approximately zero, so that $\tilde{U}(t)_{1j} = U_{ej}$

Thus

$$P(\nu_e \to \nu_e) = \sum_{j=1}^{2} |U_{ej}|^2 \, |(\tilde{U}(0))_{1j}|^2$$

$$= |U_{e1}|^2 |(\tilde{U}(0))_{11}|^2 + |U_{e2}|^2 |(\tilde{U}(0))_{12}|^2$$

Here $\tilde{U}(0) \equiv e^{-i\sigma_2 \tilde{\theta}(0)} = \begin{pmatrix} \cos\tilde{\theta}(0) & -\sin\tilde{\theta}(0) \\ \sin\tilde{\theta}(0) & \cos\tilde{\theta}(0) \end{pmatrix}$ wher $\tilde{\theta}(0)$ is the effective mixing angle at the center of the Sun

$$(\tilde{U}(0))_{12} = -\sin\tilde{\theta}(0)$$

$$U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad U_{e1} = \cos\theta, \; U_{e2} = -\sin\theta$$

So we have

$$P(\nu_e \to \nu_e) = \cos^2\theta \cos^2\tilde{\theta}(0) + \sin^2\theta \sin^2\tilde{\theta}(0)$$
$$= \frac{1}{2}(1+\cos 2\theta)\cdot\frac{1}{2}(1+\cos 2\tilde{\theta}(0)) + \frac{1}{2}(1-\cos 2\theta)\frac{1}{2}(1-\cos 2\tilde{\theta}(0))$$
$$= \frac{1}{2}\{1+\cos 2\theta \cos 2\tilde{\theta}(0)\}$$

Actually the effective mixing angle can be obtained as follows.

$$\underbrace{U}_{e^{-i\sigma_2\theta}} \underbrace{\mathrm{diag}(E_1, E_2)}_{\frac{1}{2}(E_1+E_2)\mathbb{1} - \frac{\Delta E}{2}\sigma_3} \underbrace{U^{-1}}_{e^{i\sigma_2\theta}} + \underbrace{\mathrm{diag}(A, 0)}_{=\frac{A}{2}\mathbb{1}+\frac{A}{2}\sigma_3} = \tilde{U}\,\mathrm{diag}(\tilde{E}_1, \tilde{E}_2)\,\tilde{U}^{-1}$$

$$\to -\frac{\Delta E}{2} \underbrace{e^{-i\sigma_2\theta}\sigma_3 e^{i\sigma_2\theta}}_{\sigma_3 e^{+i\sigma_2\theta} e^{i\sigma_2\theta}} + \frac{A}{2}\sigma_3 = -\frac{1}{2}\Big[\underbrace{(\Delta E\cos 2\theta - A)}_{=\alpha}\sigma_3 + \underbrace{\Delta E \sin 2\theta}_{=\beta}\sigma_1\Big]$$

$$= \sigma_3 e^{2i\sigma_2\theta}$$
$$= \sigma_3 \cos 2\theta + \sigma_1 \sin 2\theta$$

Diagonalization of the 2x2 matrix $\alpha\sigma_3 + \beta\sigma_1$ can be done by multiplying $\begin{cases} e^{+i\sigma_2\tilde{\theta}} \text{ from the left} \\ e^{-i\sigma_2\tilde{\theta}} \text{ from the right} \end{cases}$:

$$e^{+i\sigma_2\tilde{\theta}}(\alpha\sigma_3+\beta\sigma_1)e^{-i\sigma_2\tilde{\theta}} = \alpha\sigma_3 e^{-2i\sigma_2\tilde{\theta}} + \beta\sigma_1 e^{-2i\sigma_2\tilde{\theta}}$$
$$= \alpha(\sigma_3\cos 2\tilde{\theta} - \sigma_1 \sin 2\tilde{\theta})$$
$$+ \beta(\sigma_1 \cos 2\tilde{\theta} + \sigma_3 \sin 2\tilde{\theta})$$

Demanding that $\alpha \sin 2\tilde{\theta} = \beta\cos 2\tilde{\theta}$, the σ_1 terms vanish.
In this case $\tan 2\tilde{\theta} = \frac{\beta}{\alpha} = \frac{\Delta E \sin 2\theta}{\Delta E \cos 2\theta - A}$

$$\cos 2\tilde{\theta} = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}, \quad \sin 2\tilde{\theta} = \frac{\beta}{\sqrt{\alpha^2+\beta^2}}$$

$$= \sqrt{\alpha^2+\beta^2}\,\sigma_3$$

$$\therefore \alpha\sigma_3 + \beta\sigma_1 = e^{-i\sigma_2\tilde{\theta}}\sqrt{\alpha^2+\beta^2}\,\sigma_3\, e^{i\sigma_2\tilde{\theta}}$$

Thus, the value of $\cos 2\tilde{\theta}(0)$ at the solar center is given by $\cos 2\tilde{\theta}(0) = \frac{\Delta E\cos 2\theta - A(0)}{\Delta\tilde{E}(0)}$

wher $\Delta\tilde{E}(0) \equiv \sqrt{\alpha^2(0)+\beta^2(0)} \equiv [\{\Delta E \cos 2\theta - A(0)\}^2 + (\Delta E \sin 2\theta)^2]^{\frac{1}{2}}$

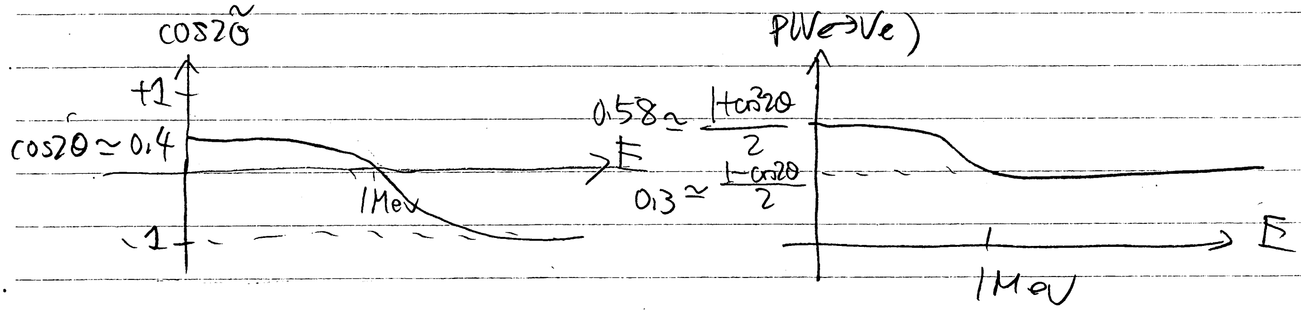
Although the probability $P(\nu_e \to \nu_e)$ does not depend on the path length (= the distance between the Sun & the Earth), it does depend on the neutrino energy through

$$\cos 2\tilde{\theta}(0) = \frac{\frac{\Delta m^2}{2E}\cos 2\theta - A(0)}{\left[\left\{\frac{\Delta m^2}{2E}\cos 2\theta - A(0)\right\}^2 + \left(\frac{\Delta m^2}{2E}\sin 2\theta\right)^2\right]^{1/2}}$$

$A(0) \equiv \sqrt{2}\, G_F N_e(0)$, $N_e(0)$ is the electron density at the Solar center and $G_F = 10^{-5}\ \mathrm{GeV}^{-2}$ is the Fermi constant.

Roughly speaking, for $E > 1\,\mathrm{MeV}$ $E \to \infty$ limit is a good approximation, so in this case $\Delta E \to 0$ $\cos 2\tilde{\theta}(0) \to -1$ (matter dominant limit)

On the other hand, for $E < 1\,\mathrm{MeV}$ $E \to 0$ limit is a good approximation so we have $\Delta E \to \infty$ $\cos 2\tilde{\theta}(0) \to \cos 2\theta \simeq 0.4$



Hence one can determine θ and Δm^2 from the mild dependence of $P(\nu_e \to \nu_e)$ on the neutrino energy.

This is in comparison with that for KamLAND ($E \sim 4\,\mathrm{MeV}$, $L \sim 200\,\mathrm{km}$)

$$P(\bar{\nu}_e \to \bar{\nu}_e) = 1 - \sin^2 2\theta \sin^2\left(\frac{\Delta m^2 L}{4E}\right)$$

which has the explicit dependence on L and E, so KamLAND has much better ability to determine Δm^2.